Newton's Method for PDE (I)

Kirsten Davis

Northern Arizona University

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What is a PDE?

A PDE is a differential equation involving partial derivatives where solutions are functions of 2 or more variables.

PDE model physical systems such as

- Heat
- Waves
- Etc.

For example, $\frac{\partial u}{\partial t} - \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0$ has solutions of the form $u(x, y, z, t)$

We are studying the PDE of the form $\Delta u + f(u) = 0$
This problem has applications in

- Star Formation
- Harvesting
- Etc.

We are interested in the boundary value problem:

- $\Delta u + su + u^3 = 0$

- $\Delta u = u_{xx} + u_{yy}$

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- $x = rcos\theta,\ y = rsin\theta$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left[cos(\theta)\frac{\partial u}{\partial x} + sin(\theta)\frac{\partial u}{\partial y}\right] = cos(\theta)\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial x \partial y}\cdot\frac{\partial y}{\partial r}\right] + sin(\theta)\left[\frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial x \partial y}\cdot\frac{\partial x}{\partial r}\right] = cos^2(\theta)\frac{\partial^2 u}{\partial x^2} + 2sin(\theta)cos(\theta)\frac{\partial^2 u}{\partial x \partial y} + sin^2(\theta)\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial r}\left[-r\cdot sin(\theta)\frac{\partial u}{\partial x} + r\cdot cos(\theta)\frac{\partial u}{\partial y}\right] = -r\cdot cos(\theta)\frac{\partial u}{\partial x} - r\cdot sin(\theta)\frac{\partial u}{\partial y} + r\left[-sin(\theta)\cdot\left(-r\cdot sin(\theta)\cdot\frac{\partial^2 u}{\partial x^2} + r\cdot cos(\theta)\frac{\partial^2 u}{\partial x \partial y}\right) + cos(\theta)\cdot\left(-r\cdot sin(\theta)\cdot\frac{\partial^2 u}{\partial x \partial y} + r\cdot cos(\theta)\frac{\partial^2 u}{\partial y^2}\right)\right] =$$

$$-r\cdot cos(\theta)\frac{\partial u}{\partial x} - r\cdot sin(\theta)\frac{\partial u}{\partial y} + r^2\left[sin^2(\theta)\cdot\frac{\partial^2 u}{\partial x^2} - 2sin(\theta)cos(\theta)\cdot\frac{\partial^2 u}{\partial x \partial y} + cos^2(\theta)\cdot\frac{\partial^2 u}{\partial y^2}\right] =$$

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$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = cos^2(\theta)\frac{\partial^2 u}{\partial x^2} + 2sin(\theta)cos(\theta)\frac{\partial^2 u}{\partial x \partial y} + sin^2(\theta)\frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\cdot\frac{\partial u}{\partial r} + sin^2(\theta)\frac{\partial^2 u}{\partial x^2} - 2sin(\theta)cos(\theta)\frac{\partial^2 u}{\partial x \partial y} + cos^2(\theta)\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\cdot\frac{\partial u}{\partial r}$$

- $\Delta u = u_{xx} + u_{yy}$
- $x = rcos\theta,\ y = rsin\theta$
$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left[cos(\theta)\frac{\partial u}{\partial x} + sin(\theta)\frac{\partial u}{\partial y}\right] = cos(\theta)\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial x \partial y}\cdot\frac{\partial y}{\partial r}\right] +$$
$$sin(\theta)\left[\frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial x \partial y}\cdot\frac{\partial x}{\partial r}\right] = cos^2(\theta)\frac{\partial^2 u}{\partial x^2} + 2sin(\theta)cos(\theta)\frac{\partial^2 u}{\partial x \partial y} + sin^2(\theta)\frac{\partial^2 u}{\partial y^2}$$
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$$r\left[-sin(\theta)\cdot\left(-r\cdot sin(\theta)\cdot\frac{\partial^2 u}{\partial x^2} + r\cdot cos(\theta)\frac{\partial^2 u}{\partial x \partial y}\right) + cos(\theta)\cdot\left(-r\cdot sin(\theta)\cdot\frac{\partial^2 u}{\partial x \partial y} + r\cdot cos(\theta)\frac{\partial^2 u}{\partial y^2}\right)\right] =$$
$$-r\cdot cos(\theta)\frac{\partial u}{\partial x} - r\cdot sin(\theta)\frac{\partial u}{\partial y} + r^2\left[sin^2(\theta)\cdot\frac{\partial^2 u}{\partial x^2} - 2sin(\theta)cos(\theta)\cdot\frac{\partial^2 u}{\partial x \partial y} + cos^2(\theta)\cdot\frac{\partial^2 u}{\partial y^2}\right] =$$
$$-r\frac{\partial u}{\partial r} + r^2\left[sin^2(\theta)\cdot\frac{\partial^2 u}{\partial x^2} - 2sin(\theta)cos(\theta)\cdot\frac{\partial^2 u}{\partial x \partial y} + cos^2(\theta)\cdot\frac{\partial^2 u}{\partial y^2}\right]$$
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = cos^2(\theta)\frac{\partial^2 u}{\partial x^2} + 2sin(\theta)cos(\theta)\frac{\partial^2 u}{\partial x \partial y} + sin^2(\theta)\frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\cdot\frac{\partial u}{\partial r} + sin^2(\theta)\frac{\partial^2 u}{\partial x^2} -$$
$$2sin(\theta)cos(\theta)\frac{\partial^2 u}{\partial x \partial y} + cos^2(\theta)\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\cdot\frac{\partial u}{\partial r}$$
- $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_\theta + u_{\theta\theta}$

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- Likewise, in function space, if a linear operator is self-adjoint then there exists an orthonormal basis of eigenfunctions that span the function space that that operator lives in.
  - Let Ψ_i be the i^{th} eigenfunction
  - Let $X = span\{\Psi_i\}$
  - $\langle \Psi_i, \Psi_j \rangle = \int \Psi_i \Psi_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

- All functions on X can be formed with linear combinations of the $\Psi_i's$
- $u \in X \Rightarrow u = \sum_{i=1}^{M} a_i \Psi_i$
- $a_j = \cdots = \langle u, \Psi_j \rangle$
- $G = span\{\Psi_1, \ldots, \Psi_M\}$

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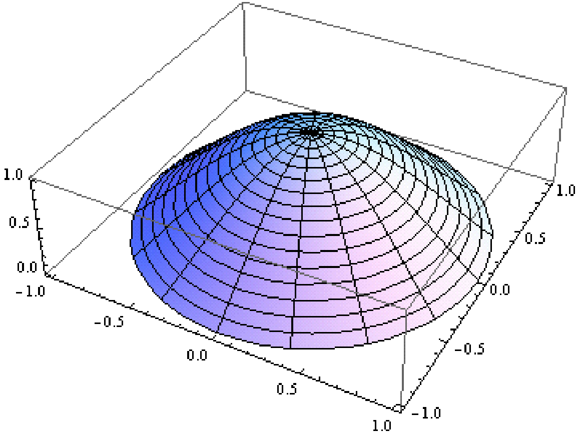
  - $-\Delta u = \lambda u$
- $-u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_\theta + u_{\theta\theta} = \lambda u$
- $z = \sqrt{(\lambda r)}$
- $z^2 u'' + z u' + (z^2 - m^2) u = 0$
  - Known as Bessel's differential equation
  - Commonly known to have certain types of solutions

- $\Psi_{0,j}(x) = J_o(z_0^k r)$
  - No θ dependence
- $\Psi_{m,j}^c(x) = J_m(z_m^k r) cos(m\theta)$
  - θ dependence with cosine
- $\Psi_{m,j}^s(x) = J_m(z_m^k r) sin(m\theta)$
  - θ dependence with sine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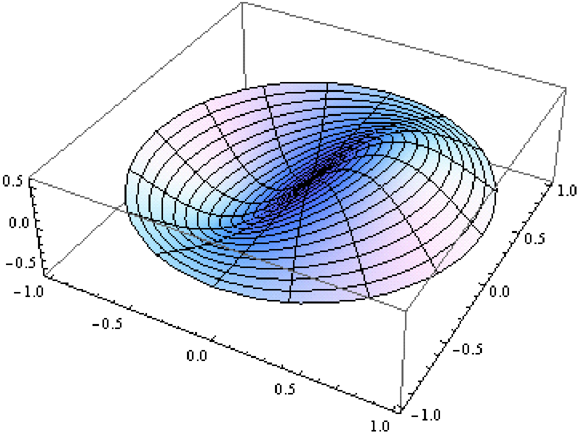
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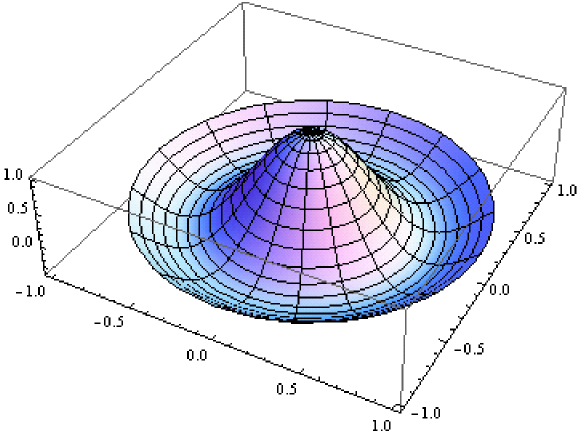
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Part (II)

- Newton's Method
- Codes
- Results
- Solution sets

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- Dr. John Neuberger, Mentor
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  - Kathleen Stigmon
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